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How Many Notions of "Sharp"?

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1. INTRODUCTION

To what extent is "sharp/unsharp" an *unsharp, ambiguous* distinction? In the framework of the *unsharp approach to quantum theory* different definitions of "sharp physical property" can be proposed. One can distinguish two basic kinds of characterizations:

(i) Purely *algebraic* definitions, which only refer to the algebraic structure of the quantum events;
(ii) *Semantic-statistical* definitions, which also refer to the relations between *events* and *states*.

In the present paper we will only discuss case (i). Case (ii) will be the subject of another paper.

Different algebraic environments give different answers to the question, "What is the structure of the quantum events?" A *minimal structure* is represented by an *effect algebra*.[4] This is a partial structure with two privileged

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[4] Effect algebras have been also called *weak ortholagebras* [6] and *unsharp ortholagebras* [2]. Equivalent structures are represented by *difference posets* ([8]).

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elements (the *certain* and the *impossible* event), where the partial operation $\boxplus$ represents an *exclusive disjunction*.

Definition 1.1. An *effect algebra* is a partial structure $\mathcal{A} = \langle A, \boxplus, \mathbf{1}, \mathbf{0}\rangle$ where $\boxplus$ is a partial binary operation on A. When $\boxplus$ is defined for a pair $a, b \in A$, we will write $\exists(a \boxplus b)$. The following conditions hold:

(i) *Conditional commutativity*:
$\exists(a \boxplus b) \Rightarrow \exists(b \boxplus a)$ and $a \boxplus b = b \boxplus a$.

(ii) *Conditional associativity*:
$[\exists(b \boxplus c)$ and $\exists(a \boxplus (b \boxplus c))] \Rightarrow [\exists(a \boxplus b)$ and $\exists((a \boxplus b) \boxplus c)$ and $a \boxplus (b \boxplus c) = (a \boxplus b) \boxplus c]$.

(iii) *Strong excluded middle*:
For any a, there exists a unique x such that $a \boxplus x = \mathbf{1}$.

(iv) *Weak consistency*:
$\exists(a \boxplus \mathbf{1}) \Rightarrow a = \mathbf{0}$.

An orthogonality relation $\perp$, a partial order relation $\sqsubseteq$, and a generalized complement $'$ can be defined in any effect algebra:

(i) $a \perp b$ iff $a \boxplus b$ is defined in A.

(ii) $a \sqsubseteq b$ iff $\exists c \in A$ such that $a \perp c$ and $b = a \boxplus c$.

(iii) The *fuzzy complement* of a is the unique element a' such that $a \boxplus a' = \mathbf{1}$ (the definition is justified by the strong excluded middle condition).

Consequently, any effect algebra gives rise to an involutive bounded poset (called also *de Morgan poset*) $\langle A, \sqsubseteq, ', \mathbf{1}, \mathbf{0}\rangle$.

Let us now consider the concrete set $E(\mathcal{H})$ of all *effects* in a Hilbert space $\mathcal{H}$. By *effect* we mean any linear bounded operator E such that $\mathbb{0} \sqsubseteq E \sqsubseteq \mathbb{1}$. The elements $\mathbb{0}$ and $\mathbb{1}$ are the null and the identity projections, respectively, while the partial order relation $\sqsubseteq$ is defined as follows: $E \sqsubseteq F$ iff for any density operator D, $\mathrm{Tr}(DE) \leq \mathrm{Tr}(DF)$ (in other words, the probability assigned by state D to the effect E is less than or equal to the probability assigned by D to the effect F).

In order to induce the structure of an effect algebra on $E(\mathcal{H})$, it is sufficient to define a partial sum $\boxplus$ as follows: $\exists(E \boxplus F)$ iff $E + F \in E(\mathcal{H})$, where $+$ is the usual sum operation. Further, $\exists(E \boxplus F) \Rightarrow E \boxplus F := E + F$. It turns out that the structure $\langle E(\mathcal{H}), \boxplus, \mathbb{1}, \mathbb{0}\rangle$ is an effect algebra, where the fuzzy complement of any effect E is $E' := \mathbb{1} - E$.

Any abstract effect algebra $\mathcal{A} = \langle A, \boxplus, \mathbf{1}, \mathbf{0}\rangle$ can be naturally extended to a total structure called a *quantum MV-algebra* (abbreviated as QMV-algebra) [5].

Both QMV-algebras and MV-algebras are total structures having the following form:

$$\mathcal{M} = (M, \oplus, ', \mathbf{1}, \mathbf{0})$$

where (i) **1**, **0** represent the certain and the impossible propositions (or events); (ii) $'$ is the negation operation; (iii) $\oplus$ represents a disjunction (*or*) which is generally nonidempotent ($a \oplus a \neq a$). A (generally nonidempotent) conjunction (*and*) is then defined via the de Morgan law: $a \odot b := (a' \oplus b')'$. On this basis, a pair consisting of an idempotent conjunction *et* ($\Cap$) and of an idempotent disjunction *vel* ($\Cup$) is then defined:

$$a \Cap b := (a \oplus b') \odot b; \qquad a \Cup b := (a \odot b') \oplus b$$

Definition 1.2. An *MV-algebra* is a structure $\mathcal{M} = (M, \oplus, ', \mathbf{1}, \mathbf{0})$, where $\oplus$ is a binary operation, $'$ is a unary operation, and **0** and **1** are special elements of M, satisfying the following axioms:

(MV1) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$
(MV2) $a \oplus \mathbf{0} = a$
(MV3) $a \oplus b = b \oplus a$
(MV4) $a \oplus \mathbf{1} = \mathbf{1}$
(MV5) $(a')' = a$
(MV6) $\mathbf{0}' = \mathbf{1}$
(MV7) $a \oplus a' = \mathbf{1}$
(MV8) $(a' \oplus b)' \oplus b = (a \oplus b')' \oplus a$

In other words, an MV-algebra represents a particular weakening of a Boolean algebra, where $\oplus$ and $\odot$ are generally nonidempotent.

A partial order relation can be defined in any MV-algebra:

$$a \leq b \quad \text{iff} \quad a \Cap b = a \tag{1.1}$$

Let us now go back to our concrete effect structure $\langle E(\mathcal{H}), \boxplus, 1, 0\rangle$. The partial operation $\boxplus$ can be extended to a total operation $\oplus$ that behaves like a *collapsed* sum. For any $E, F \in E(\mathcal{H})$

$$E \oplus F = \begin{cases} E + F & \text{if } \exists (E \boxplus F) \\ 1 & \text{otherwise} \end{cases}$$

Further, let us put $E' = \mathbb{1} - E$. The structure $\mathcal{E}(\mathcal{H}) = \langle E(\mathcal{H}), \oplus, ', 1, 0\rangle$ turns out to be "very close" to an MV-algebra. However, something is missing: $\mathcal{E}(\mathcal{H})$ satisfies the first seven axioms of our definition (MV1)–(MV7); at the same time one can easily check that the axiom (MV8) (usually called the "Łukasiewicz axiom") is violated. As a consequence, the Łukasiewicz axiom must be conveniently weakened to obtain a representation for our concrete effect structure. This can be done by means of the notion of a QMV-algebra.

Definition 1.3. A *quantum MV-algebra* (*QMV-algebra*) is a structure $\mathcal{M} = (M, \oplus, ', \mathbf{1}, \mathbf{0})$ that satisfies, besides axioms (MV1)–(MV7) of MV-algebras, the following axiom (which represents a weakening of (MV8)):

(QMV8) $a \oplus [(a' \Cap b) \Cap (c \Cap a')] = (a \oplus b) \Cap (a \oplus c)$

The operations $\Cap$ and $\Cup$ of a QMV-algebra $\mathcal{M}$ are generally noncommutative. As a consequence (differently from MV-algebra), they do not represent lattice operations. One can prove:

Theorem 1.1 [5]. A QMV-algebra $\mathcal{M}$ is an MV-algebra iff for all $a, b \in M$: $a \Cap b = b \Cap a$.

At the same time, any QMV-algebra $\mathcal{M} = (M, \oplus, ', \mathbf{1}, \mathbf{0})$ gives rise to an involutive bounded poset $\langle M, \leq, ', \mathbf{1}, \mathbf{0}\rangle$, where the partial order relation is defined as in the MV case by (1.1).

Theorem 1.2. The structure $\mathscr{E}(\mathscr{H}) = \langle E(\mathscr{H}), \oplus, ', \mathbf{1}, \mathbf{0}\rangle$ is a QMV-algebra.

A concrete effect poset, whose support is $E(\mathscr{H})$, can be naturally extended to a richer structure, equipped with a new complement $^\sim$, that has an intuitionistic-like behavior:

(**) $E^\sim$ is the projection operator $P_{Ker(E)}$ whose range is the kernel $Ker(E)$ of E, consisting of all vectors that are transformed by the operator E into the null vector.

By definition, the intuitionistic complement of an effect is always a projection. In the particular case where E is a projection, it turns out that $E' = E^\sim$. The structure $\langle E(\mathscr{H}), \sqsubseteq, ', ^\sim, \mathbf{1}, \mathbf{0}\rangle$ is a particular example of a *Brouwer–Zadeh poset* [1].

Definition 1.4. A *Brouwer–Zadeh poset* (simply a *BZ-poset*) is a structure $\langle B \sqsubseteq, ', ^\sim, \mathbf{1}, \mathbf{0}\rangle$, where

(1) $\langle B \sqsubseteq, ', \mathbf{1}, \mathbf{0}\rangle$ is a Kleene poset (i.e., a de Morgan poset s.t.: $a \sqsubseteq a'$ and $b \sqsubseteq b' \Rightarrow \mathrm{a} \sqsubseteq \mathrm{b}'$).
(2) $^\sim$ is a 1-ary operation on B that behaves like an intuitionistic complement: (i) $a \sqcap a^\sim = \mathbf{0}$; (ii) $a \sqsubseteq a^{\sim\sim}$; (iii) $a \sqsubseteq b \curvearrowright b^\sim \sqsubseteq a^\sim$ (where $\sqcap$ is the *inf*).
(3) The following relation connects the fuzzy and the intuitionistic complement: $a^{\sim'} = a^{\sim\sim}$.

We will use the following abbreviations: $\mathbb{EA}$ for the class of all effect algebras, $\mathbb{QMV}$ for the class of all QMV algebras, $\mathbb{MV}$ for the class of all MV algebras, $\mathbb{BZ}$ for the class of all BZ posets, and $\mathbb{EH}$ for the class of all QMV algebras based on a concrete $E(\mathscr{H})$.

2. ALGEBRAIC NOTIONS OF "SHARP"

Suppose a QMV-framework. We introduce six definitions of "sharp element." Let $\mathcal{M} = \langle \mathcal{M}, \oplus, ', \mathbf{1}, \mathbf{0} \rangle \in \mathbb{QMV}$ and let $a \in M$.

Definition 2.1:

(i) a is *principal* (or *sharp1*) $\Leftrightarrow \forall b, c \in M$: $b, c \leq a$ and $b \leq c' \Rightarrow b \oplus c \leq a$.
(ii) a is *Aristotle-sharp* (or *Aristotelian* or *sharp2*) $\Leftrightarrow a \sqcap a' = \mathbf{0}$ (in other words, a satisfies the noncontradiction principle).
(iii) a is *Łukasiewicz-sharp* (or *sharp3*) $\Leftrightarrow a \Cap a' = \mathbf{0}$.
(iv) a is *strongly Łukasiewicz-sharp* (or *sharp4*) $\Leftrightarrow a \sqcap a' = \mathbf{0}$ and $a \Cap a' = a' \Cap a$.
(v) a is *Boole-sharp* (or *Boolean* or *sharp5*) $\Leftrightarrow a \oplus a = a$ (in other words the sum operation is idempotent).
(vi) a is *strongly Boolean* (or *sharp6*) $\Leftrightarrow \{b \mid b \leq a\}$ is an ideal of $\mathcal{M}$.

Notice that the definitions of *principal* and *Aristotelian* element can be defined also in the case of an effect algebra.

Theorem 2.1. $\forall a \in M$: a is principal $\Rightarrow a$ is Aristotelian.

Proof. Let a be a principal element of M. Suppose $b \leq a, a'$. We want to show that $b = \mathbf{0}$. Now, $a \leq a$ and $b \leq a'$. Thus, $a \oplus b \leq a$ since a is principal. Therefore, $a \oplus b = a$. By the cancellation law, there follows $b = \mathbf{0}$. ■

Let $E_s(\mathcal{H})$ be the class of all *special* effects of a Hilbert space $\mathcal{H}$.[5] An effect E is called *special* iff either E is trivial ($\mathbb{1}$ or $\mathbb{0}$) or E satisfies the following condition: there exist two density operators D_1 and D_2 such that $\mathrm{Tr}(D_1E) < 1/2$ and $\mathrm{Tr}(D_2E) > 1/2$. It is easy to see that $E_s(\mathcal{H})$ is closed under the operation $'$.

Let us define the following operation over $E_s(\mathcal{H})$:

$$E \oplus F = \begin{cases} E + F & \text{if } E + F \in E_s(\mathcal{H}) \\ \mathbb{1} & \text{otherwise} \end{cases} \tag{2.1}$$

It turns out that the structure $\mathcal{E}_s(\mathcal{H}) = \langle E_s(H), \oplus, ', \mathbb{1}, \mathbb{0} \rangle$ is a QMV-algebra s.t. $\forall E, F \in E_s(H)$: if $E \leq F$, then $E \sqsubseteq F$, where

$$E \sqsubseteq F \quad \text{iff for any density operator} \quad D: \mathrm{Tr}(DE) \leq \mathrm{Tr}(DF) \tag{2.2}$$

Notice that the inverse relation does not generally hold.

[5] Special effects are also called *regular effects or unsharp properties* [3].

The set of all special effects is closed under the intuitionistic-like complement $^{\sim}$ defined according to (**). However, the structure $\langle E_s(\mathcal{H}), \leq, ', ^{\sim}, 1, 0\rangle$ is *not* a BZ poset since in general $E \not\leq E^{\sim\sim}$. For example, let $E := \alpha P$, where $\alpha \in (1/2, 1)$ and P is any nontrivial projection. It turns out that $E \oplus E^{\sim}$ is an effect, which is not special. Consequently: $E \not\leq E^{\sim\sim}$.

Theorem 2.2. Aristotelian $\not\Rightarrow$ principal.

Proof. Let $\mathcal{E}_s(\mathbb{C}^3)$ be the QMV-algebra of all special effects of $\mathbb{C}^3$. Let us consider the following effect:

$$E := P_1 + \tfrac{1}{6}P_2 + P_3$$

where $\{P_1, P_2, P_3\}$ is a set of pairwise orthogonal (w.r.t. the partial ordering $\sqsubseteq$) projections of $\mathbb{C}^3$. One can readily see that E is a special effect. First, we show that $E \sqcap E' = \mathbb{0}$, w.r.t. the partial ordering $\leq$. Suppose $F \leq E, E'$. Then $F \leq F'$. Thus, $F \sqsubseteq F'$, so that $F \sqsubseteq \frac{1}{2}\mathbb{1}$. Since F is special, it follows that $F = \mathbb{0}$. Accordingly, E is Aristotelian. We show now that E is not principal. Let us consider the following two effects:

$$F := \tfrac{1}{6}P_1 + \tfrac{1}{6}P_2 + \tfrac{2}{3}P_3 \qquad \text{and} \qquad G := \tfrac{5}{6}P_1 + \tfrac{1}{6}P_2 + \tfrac{1}{3}P_3$$

Clearly, $F + G = P_1 + \frac{1}{3}P_2 + P_3 \in E_s(\mathbb{C}^3)$. An easy computation shows that $F \leq G'$ and $F, G \leq E$. However,

$$F \oplus G = P_1 + \tfrac{1}{3}P_2 + P_3 \not\leq E \quad \blacksquare$$

The following theorem shows that the notions "principal" and "Aristotelian" are equivalent whenever restricted to the concrete QMV-algebra of all effects.

Theorem 2.3. $\forall E \in E(\mathcal{H})$: E is a projection $\Leftrightarrow$ E is principal $\Leftrightarrow$ E is Aristotelian.

Proof. E is a projection $\Leftrightarrow$ E is Aristotelian [9].

By Theorem 2.1, it suffices to show that if an effect E is a projection, then E is principal. Let P be any projection of $E(\mathcal{H})$ and let E, F be any two effects s.t. $E \leq F'$ and $E, F \leq P$. By definition, $E \oplus F = E + F$. By [6],

$$E = PE = EP \qquad \text{and} \qquad F = PF = FP \tag{2.3}$$

Thus,

$$(E \oplus F)P = (E + F)P = P(E \oplus F)$$

Hence, by [6], $E \oplus F \leq P$. $\blacksquare$

Theorem 2.4. Let $\mathcal{M} = \langle M, \oplus, ', \mathbf{1}, \mathbf{0}\rangle$ be a QMV-algebra. The following conditions are equivalent $\forall a \in M$:

(i) a is Łukasiewicz-sharp (sharp3).
(ii) a is strongly Łukasiewicz-sharp (sharp4).
(iii) a is Boolean (sharp5).
(iv) a is strongly Boolean (sharp6).

Proof. Conditions (iii) and (iv) are clearly equivalent.

(i) $\Rightarrow$ (ii). Suppose $a \Cap a' = \mathbf{0}$. By [5], $a' \oplus (a \Cap a') = a' \oplus a'$, so that $a' \oplus a' = a'$. Hence: $a' \Cap a = \mathbf{0} = a \Cap a'$. It remains to show that $a \sqcap a' = \mathbf{0}$. Suppose $b \leq a, a'$. By [5], $b \leq a \Cap a' = \mathbf{0}$.

(ii) $\Rightarrow$ (iii). Suppose $a' \Cap a = a \Cap a'$ and $a \sqcap a' = \mathbf{0}$. By [4], $a \Cap a' \leq a'$ and $a \Cap a' = a' \Cap a \leq a$. Thus, $a \Cap a' = \mathbf{0}$. By [5], it follows that $a \oplus a = a$.

(iii) $\Rightarrow$ (i). Straightforward. ■

Theorem 2.5. Let $\mathcal{M}$ be a QMV-algebra. The following property holds $\forall a \in M$: a is Łukasiewicz (or strongly Łukasiewicz or Boolean or strongly Boolean) $\Rightarrow$ a is principal.

Proof. It follows from the monotonicity of $\oplus$. ■

The following theorem shows that a principal element need not be Boolean (equivalently, Łukasiewicz, strongly Łukasiewicz, strongly Boolean).

Theorem 2.6. Any nontrivial projection in $E(\mathcal{H})$ is principal, but not Boolean.

Proof. Let P be any nontrivial projection. Then, by Theorem 2.3, P is principal. However (by definition of $\oplus$), $P \oplus P = \mathbb{1} \neq P$. ■

Theorem 2.7. The six notions of "sharp" collapse in the case of MV-algebras.

Proof. Every MV-algebra is a QMV-algebra. Thus, by Theorem 2.1 and Theorem 2.4, it is sufficient to show that every Aristotelian element is Łukasiewicz-sharp. In every MV-algebra the operation $\Cap$ coincides with the *inf*. Accordingly, if $a \sqcap a' = \mathbf{0}$, then also $a \Cap a' = \mathbf{0}$. ■

Summing up:

$\mathbb{QMV}$

Łukasiewicz $\Leftrightarrow$ Strong Łukasiewicz $\Leftrightarrow$ Boole $\Leftrightarrow$ Strong Boole

$\Downarrow\Uparrow$

Principal $\Rightarrow$ Aristotle

$\not\Leftarrow$

$\mathbb{EH}$

Łukasiewicz ⇔ Strong Łukasiewicz ⇔ Boole ⇔ Strong Boole
⇓ ⇑
Principal ⇔ Aristotle

$\mathbb{MV}$

Principal ⇔ Aristotle ⇔ Łukasiewicz
⇕ ⇕
Strong Łukasiewicz ⇔ Boole ⇔ Strong Boole

Let us now refer to effect algebras that are also BZ posets (BZ effect algebras). This permits us to define a new notion of sharpness:

Definition 2.2. An element a is called *Brouwer-sharp* (or *Brouwerian* or *sharp7*) $\Leftrightarrow a = a^{\sim\sim}$.

Theorem 2.8. Brouwer ⇒ Aristotle.

Proof. Suppose a is Brouwer-sharp. Then $a' = a^{\sim\sim\prime} = a^{\sim\sim\sim} = a^{\sim}$. Let b be any element s.t. $b \sqsubseteq a, a'$. Since $a' = a^{\sim}$, there follows $b \sqsubseteq a \sqcap a^{\sim} = \mathbf{0}$. ■

It should be noticed that every effect algebra $\mathcal{A} = \langle A, \boxplus, \mathbf{1}, \mathbf{0}\rangle$ can be trivially organized into a BZ effect algebra. It is sufficient to define the operation $^{\sim}$ in the following way:

$$a^{\sim} = \begin{cases} \mathbf{1} & \text{if} \quad a = \mathbf{0} \\ \mathbf{0} & \text{otherwise} \end{cases}$$

The structure $\mathcal{A} = \langle A, \boxplus, {}^{\sim}, \mathbf{1}, \mathbf{0}\rangle$ turns out to be a BZ effect algebra. In this case, the class of all Brouwerian elements contains only $\mathbf{0}$ and $\mathbf{1}$.

Theorem 2.9. Aristotle ⇏ Brouwer.

Proof. Let $\mathcal{A}$ be any *orthoalgebra* containing more than two elements. We recall that an orthoalgebra is an effect algebra where every element is Aristotelian. Equip $\mathcal{A}$ with the trivial Brouwer complement. Thus, every nontrivial element of $\mathcal{A}$ is Aristotelian but non-Brouwerian. ■

Gudder [7] has given a characterization of the class of effect algebras that give rise to BZ effect algebras where the notion of Brouwerian and Aristotelian sharp element coincide.

Theorem 2.10. Let $E \in E(\mathcal{H})$: E is Aristotelian ⇔ E is Brouwerian.

Proof. Straightforward. ■

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